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On cohesion stable graphs*

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Abstract

The cohesion of a graph was introduced to model vulnerability of a graph relative to the neighborhoods of its vertices. We are concerned in this paper with the changes in this parameter when an edge is deleted. In particular, after displaying some results on stability under edge destruction, we go on to display various infinite classes of cohesion stable graphs. Several ways in which graphs or parts of graphs may be combined to produce stable graphs are also presented, along with a look at what cannot be stated at this time.

1 Definitions and early results

In this paper we further take up the study of the cohesion parameter for a graph. In this case we are particularly interested in graphs which do not change cohesion when any edge of the graph is deleted. The cohesion concept was first introduced in [3] in order to distinguish vertices which are in a particularly vulnerable situation relative to nearness to being a cutpoint in alliance graphs. In [6] the authors first began to examine the effect on the parameter when a graph changes by examining new graphs formed by the addition of edges. If the concept is to have applicability to a wider variety of problems, then the obvious way in which to change the graph is by the deletion of edges. For example, if one is to monitor certain vertices in a graph, then it could be important to know which vertices could become cutvertices with only a few edge failures. For a given graph, the average number of edge failures necesssary for vertices to become cutvertices seems well worth investigation. The cohesion of a graph is a graph-theoretical concept which attempts to look at graphs in this 'average' kind of way.

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Various studies of connectivity parameters have been undertaken in this regard with cohesion fitting best those involving the so-called 'mixed' parameters; i.e. those which destroy a graph by a combination of both vertices and edges.

We will make only those definitions here which are necessary to this paper. All graph theory notations will follow [1] and further information on cohesion can be gained by examining [4–8]. In particular, for a subset of vertices S in graph G, $\langle S \rangle$ will denote the subgraph induced by S.

Definition 1.1. The *cohesion of a vertex* in a graph is zero if the vertex is a cutvertex or an isolated vertex (by convention) and is one less than the degree of its neighbor if it is a pendant vertex. In all other cases the *cohesion of a vertex* x, denoted $\mu(x)$, is the minimum number of edges whose deletion causes x to be cutvertex in the resultant graph. The set of edges whose deletion makes x a cutvertex is called a *cohesion set* for x. This set is minimum but not necessarily unique. If this set of edges consists of all the edges incident with a neighbor v of x, except the edge vx, the set is called a *neighborhood cohesion set with center* v.

The minimum number of edges is designed to indicate how close a vertex is to being a cutvertex of the graph. In the case of a pendant vertex v, the removal of all the edges incident with the neighbor save the one with v produces a K_2 which has vertex connectivity one and both vertices are considered cutvertices. An alternative way of looking at the cohesion of a vertex with degree greater than or equal to two is given by the following remark from [3].

Remark 1.2. If $p(x,y)$ denotes the maximum number of edge disjoint paths between two vertices x and y, then the cohesion of a vertex v is the minimum $p(x,y)$ in $G-v$, where the minimum is taken over all pairs of neighbors of v.

There is more than one way to make cohesion a global graph parameter. For the purposes of this paper, wherein we are concerned with edge deletion, we use the idea of 'average' cohesion for the cohesion of a graph. Since a given graph has a fixed number of vertices which does not change with edge deletion, there is no need to divide by this number when defining the cohesion.

Definition 1.3. The *cohesion of a graph* G, denoted $\mu(G)$, is the sum of the cohesions of its vertices.

Edges whose deletion have no effect on the cohesion of a graph are interesting ones. Graphs where all such edges are of this kind are the subject of this paper.

Definition 1.4. An edge e in a graph G is called *s-stable* if the cohesion of $G-e$ is the same as the cohesion of G. An edge whose deletion changes the cohesion of no vertex is called *stable*.

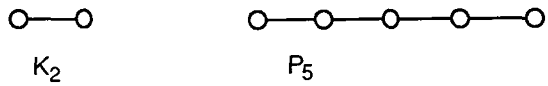


Fig. 1. The stable trees.

Of course, stable edges are always s-stable as well. We will not be concerned specifically with stable edges here, although they are also very important; for some interesting results on constructing graphs with a large number of such edges the reader is referred to [4]. We are now ready to give the basic definition for this paper.

Definition 1.5. A graph G is called *stable* if each of its edges is s-stable.

Notation. If G is a graph and e is an edge of G, then the cohesion of a vertex v in $G' = G - e$ will be denoted by $\mu'(v)$.

The question of looking for stable graphs was taken up in [7], where we first constructed graphs with large numbers of s-stable edges and which were 'asymptotically' stable. That earlier paper has an erroneous class of stable graphs which was corrected in the next issue of the same journal [8]. The present paper takes a much more thorough look at stable graphs and explains the examples which were only highlighted in the earlier paper.

It is interesting to begin the quest for stable graphs by examining trees. A straightforward counting argument will show that any tree which has a vertex with two or more pendant edges is not stable (delete a pendant edge), thus narrowing the search for stable trees considerably. Similarly, one can see that vertices of degree more than two cannot exist in such a graph. Hence, the following proposition is a first indication that such graphs may indeed exist. A quick examination of Fig. 1 will help one to understand the concept of stable graphs.

Proposition 1.6. *The only stable trees are K_2 and P_5.*

2. Looking for nontrivial stable graphs

To show that a graph is not stable, one needs to find an edge whose deletion changes the cohesion of the graph. It is relatively easy to show that when an edge is deleted from a graph the only vertices whose cohesions may increase are those which are incident with the deleted edge. (See [7] for the details of this result.)

The following proposition gives some indication as to how to discern nonstable graphs and is useful in the sequel.

Proposition 2.1. *Suppose v is not a cutvertex. A vertex v of degree three or more, incident with an edge $e=uv$, increases in cohesion when e is deleted if and only if e is a bridge in $G-U$ for every cohesion set U of v.*

Proof. We prove the contrapositive in each direction. Suppose that e is not a bridge in $G-U$ for some cohesion set U of v in G. Clearly, $G-v-U$ has exactly two components, each of which contains at least one neighbour of v. If u is in a component C, then C also contains another neighbor z of v; otherwise uv would be a bridge in $G-U$. Since $G'-v-U=G-v-U$ and z remains a neighbor of v in G', we may conclude that U separates z from other neighbors of v in G'. By Remark 1.2, then, we have that $\mu'(v)\leqslant\mu(v)$.

Conversely, suppose $\mu'(v)\leqslant\mu(v)$. As we discussed earlier and is proven in [7], we must have $\mu'(v)\geqslant\mu(v)$. Suppose that equality holds and let U^* be a cohesion set for v in G'. So U^* separates two neighbors of v in $G'-v$, z and w, and neither of these neighbors is u, i.e. $G'-v-U^*$ has exactly two components with z in one and w in another. However, u must share a component with one of the other two vertices and we assume it to be z. So, in $G-U^*$, there is a path from u to z disjoint from vu and zv. Thus, uv is on a cycle and hence is not a bridge in $G-U^*$. □

The next theorem is useful in discovering that a graph is not stable. A *clique* is a complete induced subgraph of a graph and a *simplicial vertex* v is a vertex for which $\langle N(v)\rangle$ is a clique, where $N(v)$ is the set of vertices adjacent to v. We are now prepared to state and prove Theorem 2.2.

Theorem 2.2. *Let G be a graph. If a maximal clique M of G with $|M|\geqslant 3$ has at least two simplicial vertices, then G is not stable.*

Proof. If a component of G is a complete graph, then G is not stable. Suppose a component of G contains a maximal clique M as in the theorem. Let $|M|=m$ and let S be the set of simplicial vertices of M and S^* the other vertices in M. If $|S|=s$ and $|S^*|=s'$, note that $s\geqslant 2$ and $s'\geqslant 1$. The cohesion of any vertex in S is $m-2$, since any other vertex in S is the center of a neighborhood cohesion set, while there must exist at least $m-2$ edge disjoint paths between any two vertices in a complete graph on m vertices and Remark 1.2 applies.

Let $e=uv$, where $u\in S$ and $v\in S^*$ and form $G'=G-e$. Then $\mu'(x)=m-3$ for all $x\in S$, $x\neq u$, since u is the center of a neighborhood cohesion set in G', and thus the cohesion of G' will have a negative input of $s-1$. So there must be a corresponding increase in order for G to be stable and increases can only occur at u or v. Since u is still adjacent to other vertices of S, its cohesion cannot increase. We now examine v. Note that $\mu(v)\leqslant m-2$ since vertices of S have degree $m-1$ and are adjacent to v.

If $\mu(v)<m-2$, then no two members of S are separated by any cohesion set of v and it is impossible for e to be a bridge when a cohesion set of v is removed, and $\mu'(v)=\mu(v)$ by Proposition 2.1. Suppose $\mu(v)=m-2$. If $m>3$, then there are at least two other vertices, say $x \in S$ and $w \in S^*$. Now x is the center of a neighborhood cohesion set U for v in G and the edges uv, vw, and wu form a cycle in $G-U$. Thus, by Proposition 2.1, $\mu'(v)=\mu(v)$. If $m=3$, then $\mu(u)=\mu(x)=1$ for $S=\{u,x\}$ and $\mu(v)=0$. Then clearly $\mu'(x)=0$, while $\mu'(u)=1$ and $\mu'(v)=0$ and G is not stable. □

3. Some stable graphs

Theorem 2.2 rules out many graphs as candidates for stable graphs. Unfortunately, the approach there does not yield a characterization of stable graphs because it is not true that every nonstable graph has an edge whose deletion decreases $\mu(G)$. There exist graphs for which the cohesion either increases or remains the same when any edge is removed. Such graphs are called *superstable* and several such are displayed in [5]. This last fact leads one to believe that stable graphs may be rare indeed.

In experimenting with a superstable graph, our first nontrivial stable graph was discovered and is pictured in Fig. 2, where the label K_6-E means a K_6 from which a 1-factor or perfect matching has been removed. A *perfect matching* is a set of pairwise nonadjacent edges which together are incident with every vertex in the graph. We will explain this graph carefully.

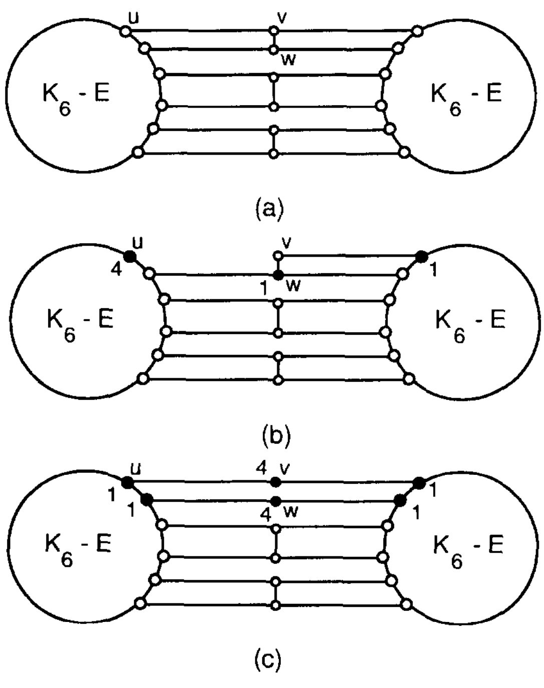


Fig. 2. A stable graph.

Throughout this discussion we will use the phrase 'draws its cohesion from w' when referring to a vertex which has as cohesion set a neighborhood cohesion set with center w. The graph has 18 vertices and 39 edges, all cohesions are two and $\mu(G)=36$. All the vertices of the graph have neighborhood cohesion sets at one of the vertices of degree three (for a discussion of such 'cohesion galaxies' see [4]). Removing an edge e from $\langle K_6-E\rangle$ results in two vertices decreasing their degree to four, so for a vertex such as u in Fig. 2(a) no new, smaller cohesion set is formed. Also, in $G-u$, $\langle K_6-E-u\rangle$ is at least 2-edge connected. Hence each pair of neighbors of u in K_6-E has at least two edge disjoint paths between them and by Remark 1.2, $\mu'(u)\geqslant 2$. Now u is still adjacent to the degree three vertex v in G' giving $\mu'(u)\leqslant 2$ and hence $\mu'(u)=\mu(u)$. Such is true for every vertex in $\langle K_6-E\rangle$. Inspection shows that no other vertex of G is affected by the deletion of e. Thus, the edges in the subgraph $\langle K_6-E\rangle$ are stable. Removal of edge uv results in only three vertices changing cohesion as shown in Fig. 2(b), where those that do change are darkened and labelled. The cohesion of the vertex u increases to four as it draws cohesion from a neighbor of degree five in the subgraph $\langle K_6-E\rangle$. On the other hand, v continues to have cohesion two because it is still adjacent to the degree three vertex w. Both vertices which remain adjacent to v drop one in cohesion as uv is in a cohesion set for each. Thus the increase in cohesion of two for u is offset by two vertices each decreasing in cohesion by one and uv is s-stable. Together v and w are centers of neighborhood cohesion sets for six vertices including themselves so it is not surprising that six vertices change cohesion when vw is deleted (Fig. 2c). The four other vertices decrease cohesion by one because vw is in a cohesion set for each; vertices v and w must draw their cohesion from the two degree five vertices instead of each other so they increase cohesion to four. The loss of four is compensated by an increase of four and the edge vw is s-stable. Notice that the consequences of edge deletion in this particular graph are localized as opposed to other graphs where many vertices change cohesion.

One extension of this example is found by employing the concept of critically n-edge connected graphs which are discussed in [2]. A graph H is said to be *critically n-edge connected* if the edge connectivity $\lambda(H)=n$ and $\lambda(H-v)=n-1$ for every vertex v of H. If we replace both $\langle K_6-E\rangle$ subgraphs with any critically 4-edge connected, 4-regular graph H with an even number of vertices and then insert the corresponding edges between them to form G, then G is stable. The condition that the graph $\langle H\rangle$ be critically 4-edge connected and 4-regular forces the vertices in $\langle H\rangle$ to draw their cohesions from the degree three vertices. If an edge e in $\langle H\rangle$ is deleted from G to form G', each vertex v of $\langle H\rangle$ still has cohesion two as $\langle H-v\rangle$ is at least 2-edge connected in $G'-v$ as in the original example. No other vertices in G are affected by the deletion of e and the edge is stable. The other edges are s-stable and behave as before. An extension on eight vertices is shown in Fig. 3.

An infinite class of stable graphs can be constructed from the stable graph of Fig. 2(a). Throughout the sequel, we use the terminology 'identify vertex u in graph G_1 with vertex v in graph G_2.' What is meant by this is that a new graph G is formed

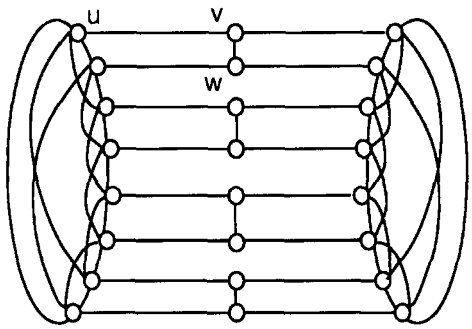


Fig. 3. A stable graph on twenty-four vertices.

where vertex u is superimposed on vertex v and edges which were incident with u or v are now incident with this new 'super' vertex.

Theorem 3.1. *For each positive integer $k \geqslant 1$, there is a 3-connected stable graph with $18k$ vertices.*

Proof. For $k=1$, the 3-connected graph in Fig. 2a is stable with 18 vertices and 39 edges. Let $k \geqslant 2$ and let $t=k-1$. To form the new stable graph link $t \geqslant 1$ copies of the graph in Fig. 4a, each to the left of the other by identifying the vertices a, b, c, d, e and f in the right-most copy with u, v, w, x, y and z, respectively, in the left-most copy. Then identify the vertices a, b, c, d, e and f on the left end of the graph with u, v, w, x, y and z, respectively, in the graph of Fig. 4b and identify the vertices u, v, w, x, y and z on the right end of the graph with a, b, c, d, e and f, respectively, in Fig. 4c. The new graph has $18k$ vertices and $39k$ edges. The edges incident with any of the lettered vertices are s-stable while all others are stable. □

We now present an example which leads to additional stable graphs which have cutvertices in them. The graph in Fig. 5a has twelve vertices, eighteen edges, and each vertex has cohesion one and thus $\mu(G)=12$. There are ten stable edges and eight s-stable edges (the darkened ones). Unlike the preceding graphs, not all of the cohesion sets are neighborhood cohesion sets as Fig. 5b demonstrates. We will see that this has important consequences. Fig. 5b–d indicate how the cohesions of the vertices change when a particular s-stable edge is deleted. The darkened vertices indicate those which change cohesion. This graph is two-edge connected with diameter five.

Paralleling the second extension shown in Fig. 4a, we obtain another large diameter stable graph, i.e. Fig. 6. In this case, note that we would have $8k+12$ vertices for $k \geqslant 0$, which is $4w$, where w is odd with $w \geqslant 3$.

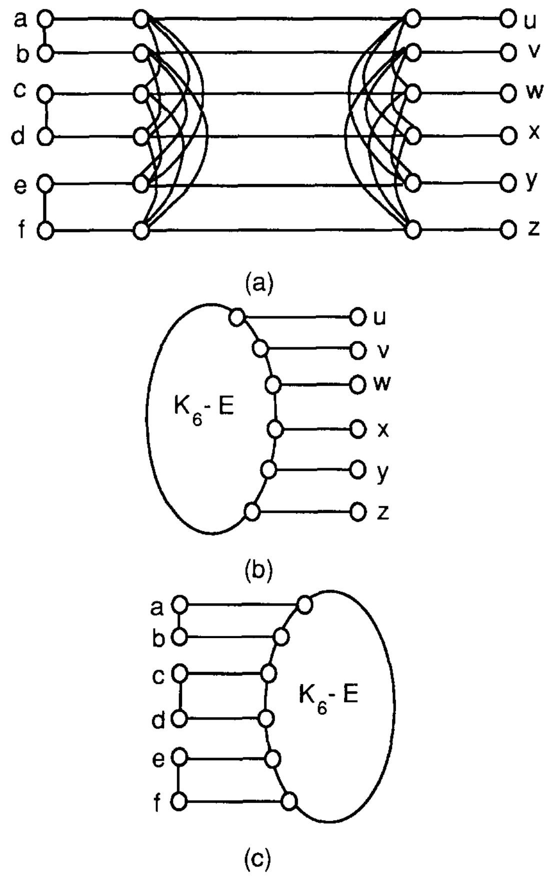


Fig. 4. The building blocks of an infinite class of stable graphs.

Theorem 3.2. *Let w be odd, $w \geqslant 3$. There exists a 2-connected stable graph on $4w$ vertices.*

In addition to its usefulness in building an infinite class of stable graphs, the graph of Fig. 5a allows a cutvertex to be introduced to form a new graph without destroying the stability property. The edges e_1 and e_2 in Fig. 7a are replaced by a cutvertex and the four edges e_5, e_6, e_7, and e_8 in Fig. 7b. All the cohesion sets behave as before except that the deletion of edge e_7 replaces deletion of edge e_2 as the cohesion set of vertex u as shown in Fig. 7c. The other degree four vertices in Fig. 7a behave similarly. The new graph now has thirteen vertices, twenty edges and each of the original vertices has the same cohesion as in the previous graph, and the new vertex has cohesion zero.

Such a construction does not always yield a stable graph. Applying a similar construction to the two-edge cutset e_3, e_4 in Fig. 7a yields the graph G in Fig. 8a. This new graph has all cohesions one except for the cutvertex and $\mu(G) = 12$, yet the edge

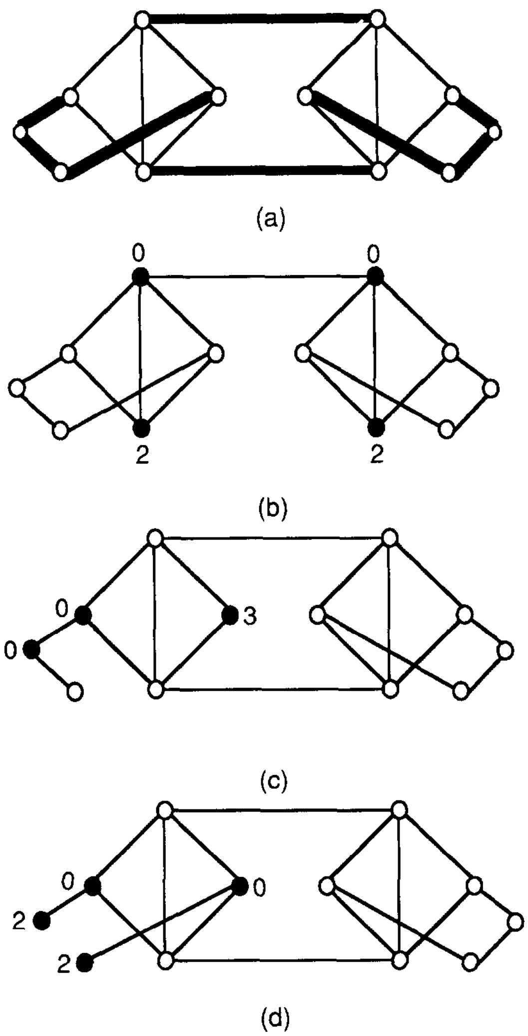


Fig. 5. Another stable graph.

e is not s-stable. The cohesion of the graph in Fig. 8b is fourteen where the darkened vertices are the only two which change cohesion. Notice that the inserted cutvertex creates a triangle with two simplicial vertices which implies by Theorem 2.2 that G is not stable.

4. Constructing stable graphs from stable graphs

What is important about stable graphs with cutvertices is that a cutvertex does not have a cohesion set. Hence, stable graphs (except K_2 and P_5) can be joined at

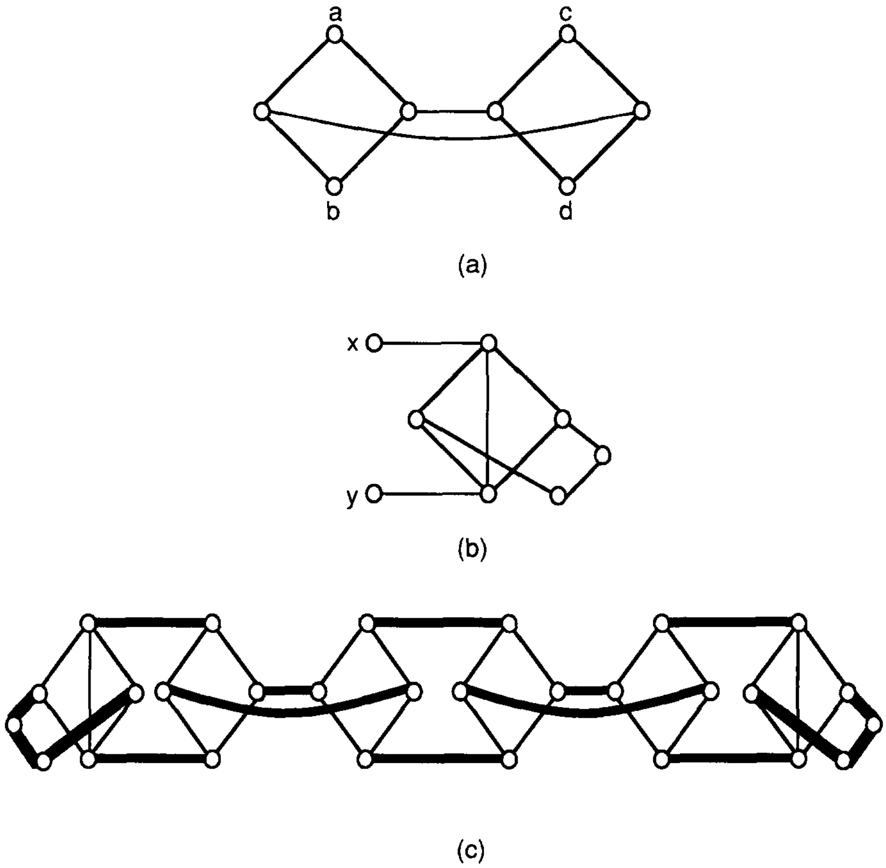


Fig. 6. An extension of the previous stable graph.

cutvertices to create new stable graphs. Before proceeding with a useful lemma, we state a remark from [5] and a corollary of it.

Remark 4.1. If G is a graph with blocks B_i, $i=1,2,\ldots,k$, then $\mu(G)=\sum\mu^*_{(G)}(B_i)$, $i=1,2,\ldots,k$, where $\mu^*_{(G)}(B_i)$ is the sum of the cohesions of all the vertices of B_i which are not cutvertices.

Corollary 4.2. *If B is a block in a stable graph G which has no pendant vertices, then $\mu^*_{(G-e)}(B-e)=\mu^*_{(G)}(B)$, for any edge $e\in E(B)$.*

We now move toward a theorem which allows us to join blocks from different stable graphs to create new stable graphs.

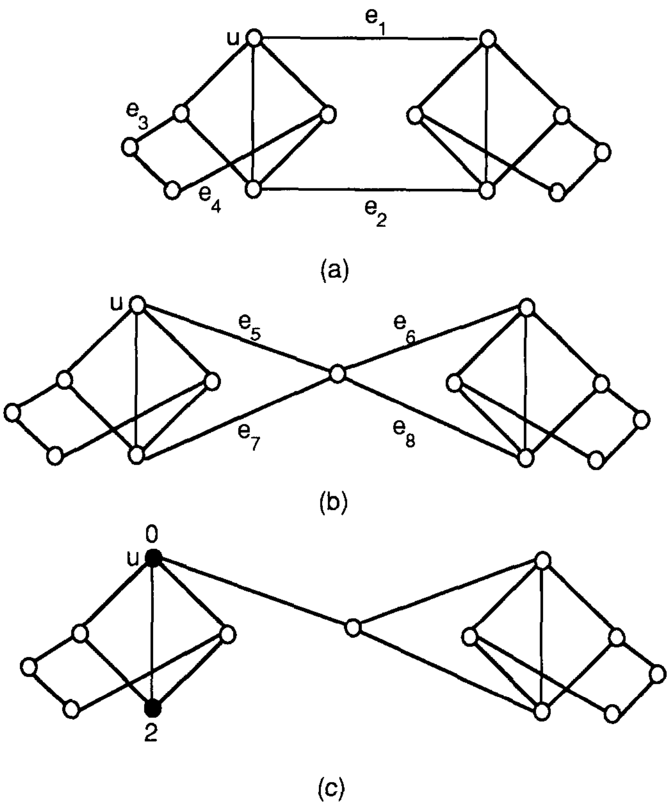


Fig. 7. A stable graph with a cutvertex.

Lemma 4.3. *Let B be a block of a stable graph G which has no endvertices and let $x_1, x_2, \ldots, x_t$, $t \geqslant 1$, be the cutvertices of B in G. If G' is another graph which has no endvertices and B' is a block in G' isomorphic with B so that the vertices in B' corresponding to x_i, $i = 1, 2, \ldots, t$, are cutvertices and no other vertices of B' are cutvertices, then each edge of B' is s-stable in G'.*

Proof. Clearly, the cohesion set of a vertex v with positive cohesion is contained within the same block as v. Thus $\mu^*_{(G')}(B') = \mu^*_{(G)}(B)$ and simple calculation yields the desired result. □

Theorem 4.4. *Let G_1 and G_2 be stable graphs with no endvertices and $\kappa(G_1) = \kappa(G_2) = 1$. If u and v are cutvertices of G_1 and G_2, respectively, then the graph G formed by adding the edge $e = uv$ between G_1 and G_2 and the graph H formed by identifying u with v are stable graphs.*

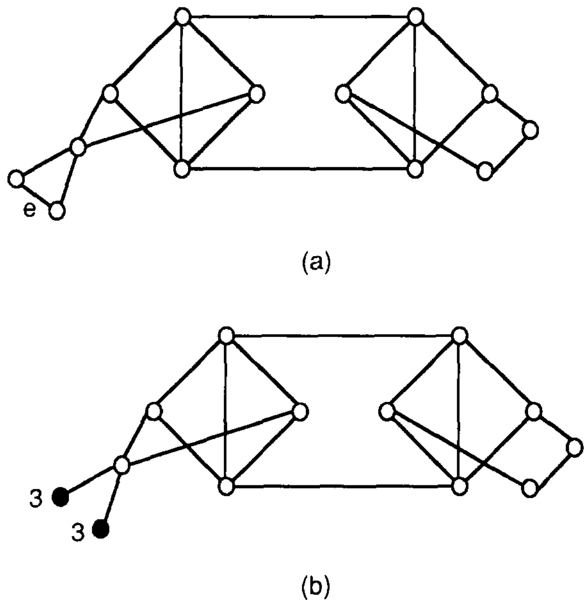


Fig. 8. The construction does not guarantee a stable graph.

Proof. We first consider G and note that e is a bridge in G. Let B_i, $i=1,2,\ldots,k$, be the blocks of G where B_1 is the K_2 block containing e. Any edge in G which is not e is s-stable by Lemma 4.3. The edge e is not in a cohesion set for any vertex in G. Furthermore, both u and v remain cutvertices when e is deleted and hence e is stable and thus G is stable.

Lemma 4.3 yields that H is stable in a similar manner. □

Fig. 9 illustrates these two methods for combining two copies of the graph in Fig. 7b. In the first graph, a bridge has been added between the cutvertices. All cohesions are one except for the cutvertices and $\mu(G)=24$. The second graph is formed by identifying the cutvertices; this graph also has cohesion twenty-four.

Consider the graph of Fig. 10; it is formed by combining only the blocks from the graph in Fig. 7b and is a stable graph. This illustrates that the blocks of stable graphs may be used to build a new stable graph even though these blocks may not be stable as graphs. Once again, the important things are that a cutvertex does not have a cohesion set and that cohesion sets are contained within blocks.

The proof to the following theorem parallels that of earlier results.

Theorem 4.5. *Let B_i, $i=1,2,\ldots,k$, be a block from a stable graph G_i which contains no endvertices. Form a graph G, which has blocks B_i such that any cutvertex of B_i in G_i is*

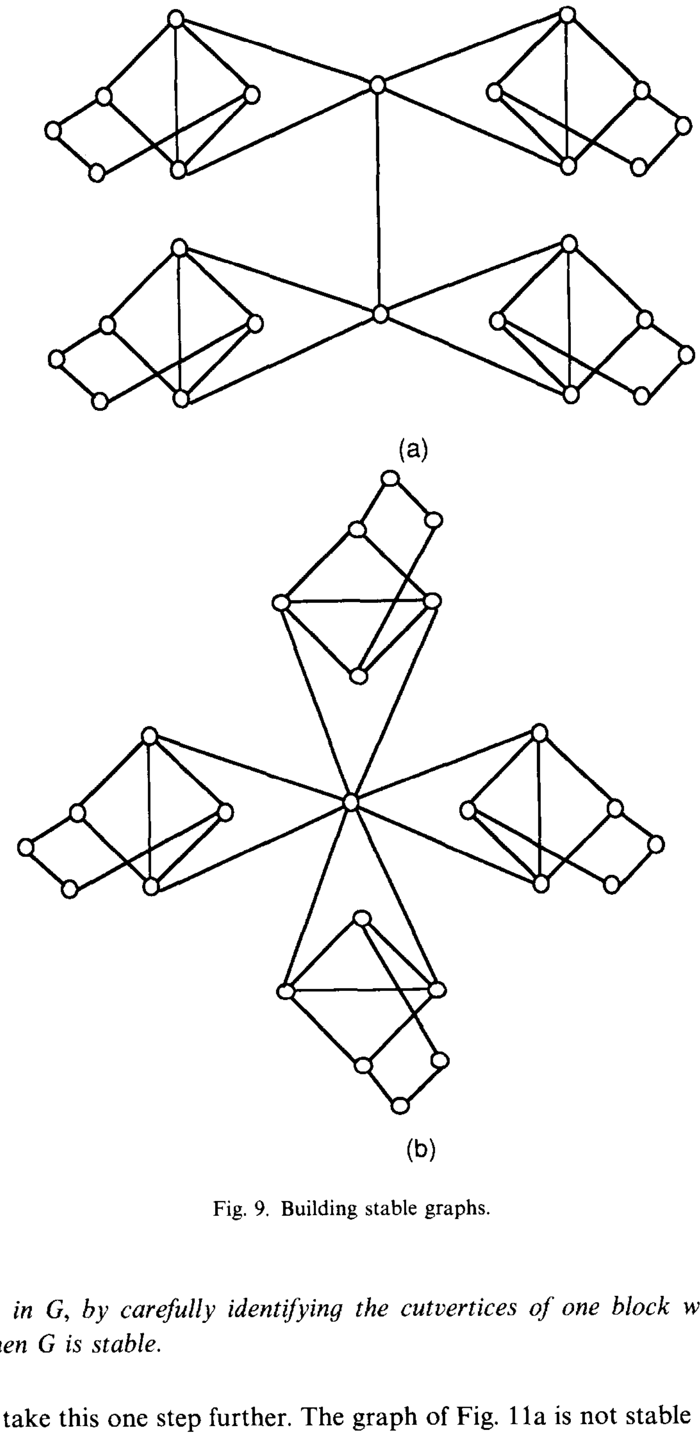


Fig. 9. Building stable graphs.

a cutvertex in G, by carefully identifying the cutvertices of one block with those of another. Then G is stable.

We now take this one step further. The graph of Fig. 11a is not stable and, in fact, none of its edges is s-stable. However, by adding our previously illustrated block from

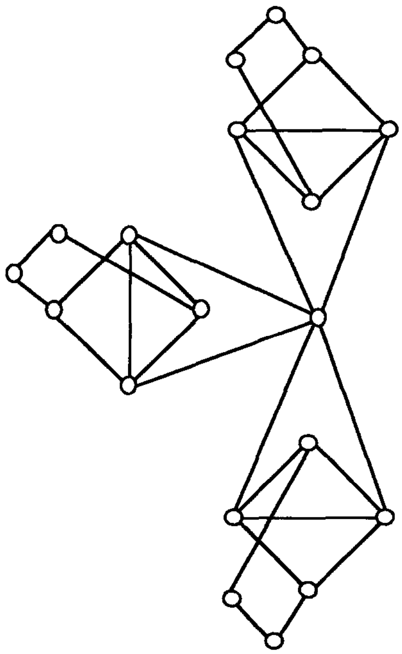


Fig. 10. Blocks of a stable graph form a new stable graph.

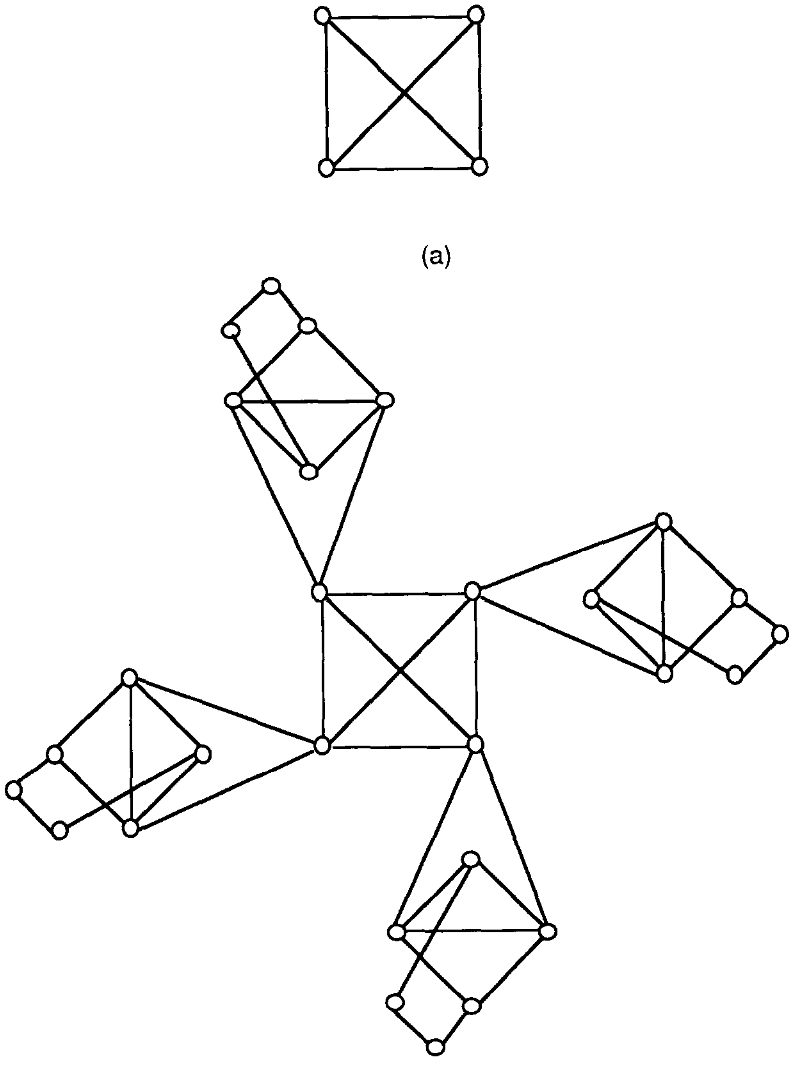
a stable graph in the manner indicated in Fig. 11b, the original graph has all edges stable and the new graph is stable. This idea is stated formally in the following theorem.

Theorem 4.6. *Any connected graph G which has no endvertices is an induced subgraph of a stable graph.*

Proof. Let H be an endblock (only one cutvertex) of a stable graph and let G be a graph of order n with no endvertices. Form the graph G' from n copies of the graph H and one copy of the graph G by identifying the cutvertex of H in each copy with a different vertex of G. Then every vertex of G is a cutvertex of G' and the edges of G are all stable in G'. By the lemma the edges in each copy of H are all s-stable in G'. □

The condition that G have no endvertices guarantees that after the removal of an edge adjacent to two cutvertices their cohesions are still zero.

Although we have constructed stable graphs with cutvertices, it is not true that stable graphs can be constructed by using all blocks which are themselves stable as graphs, as the following result from [7] illustrates.



(a)

(b)

Fig. 11. Making an arbitrary graph stable.

Theorem 4.7. *If a nonblock graph G has an endblock (not equal to K_2) which is stable when considered as a graph, then G is not stable.*

A nonendblock of a stable graph can be stable as a graph. For instance, let G be the stable graph in Fig. 12a, which is a reproduction of Fig. 2a, and let H be the block in Fig. 12b where the vertex v is the cutvertex. Using the construction in Theorem 4.6, a stable graph with G as a stable block is obtained.

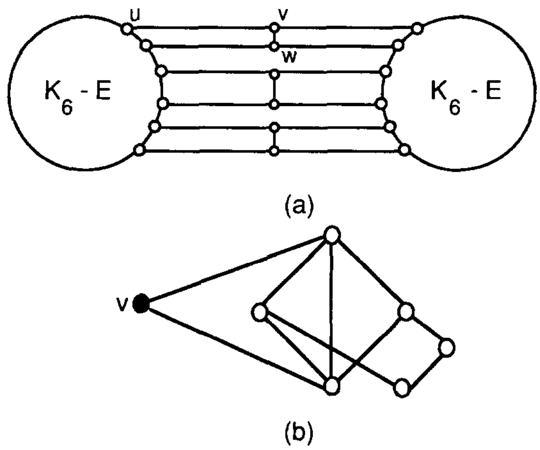


Fig. 12. Blocks which form a stable graph with a stable block.

At present, we have no examples of stable graphs without cutvertices other than those based on the graphs of Figs. 2, 3, and 5.

It is interesting to note that in each example of a 2-connected stable graph all vertices have the same cohesion. It is not known if this is always true. In fact, no sufficient conditions have been found for a graph to be stable. The difficulty arises when one tries to examine the s-stable edges which are not stable. Such an edge's behavior does not appear to be predictable and hence makes it difficult to analyze the cohesion of the graph.

In closing, we note that a computer survey done by Prof. Mark Ellingham of Vanderbilt (private communication) has revealed that there are no stable graphs with nine or fewer vertices, except for the two trees, K_2 and P_5.

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